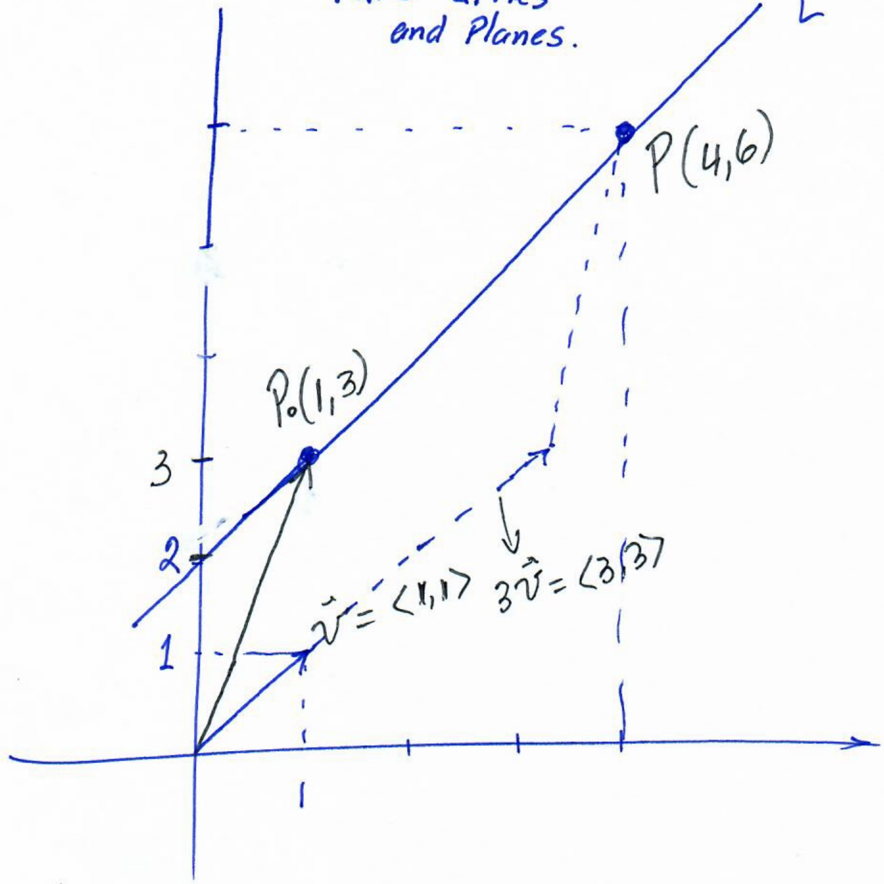
# 12.5 Lines and Planes.






$$L: \langle x, y\ z\rangle = t\langle 1,1\rangle + \langle 1,3\rangle$$

Line passing through points $\overset{(x_1,y_1)}{(1,3)}$ and $\overset{(x_2,y_2)}{(4,6)}$

<u>Slope</u> $m = \dfrac{6-3}{4-1} = \dfrac{y_2-y_1}{x_2-x_1} = \dfrac{3}{3} = 1$

Line $y-3 = 1(x-1)$ or $\boxed{y = x+2}$

This derivation of the equation of a line in the plane cannot be extended to the space.

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<u>Alternative:</u> ① Find a vector in the same direction of the line $L$

Show this geometrically

$\vec{v} = \langle 4,6\rangle - \langle 1,3\rangle = \langle 1,1\rangle.$

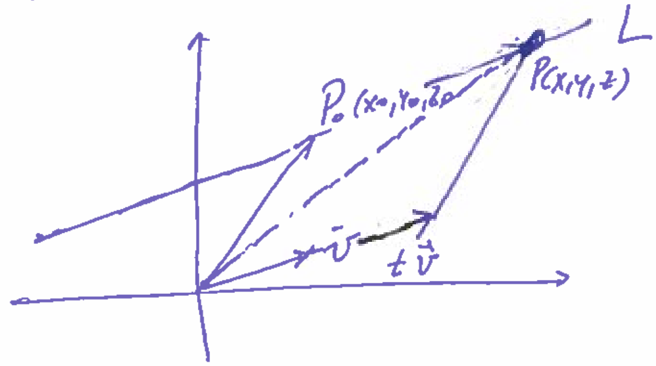
② obtain points in $L$ by adding a multiple of $\vec{v}$ $t\vec{v}$ to the position vector $\langle 1,3\rangle$ of $P_0(1,3)$

Any $(x,y) \in L$ satisfies $\langle x,y\rangle = t\langle 1,1\rangle + \langle 1,3\rangle$.

# 12.5 Lines and Planes.

## Vector form of a Line that passes through $P_0(x_0,y_0,z_0)$ with direction $\vec{v}=\langle a,b,c\rangle$

To every point $P(x,y,z)$ on $L$ there is a vector $\langle x,y,z\rangle$ associated




Clearly, by adding vectors

$$\boxed{\langle x,y,z\rangle = t\langle a,b,c\rangle + \langle x_0,y_0,z_0\rangle}, \quad t\in\mathbb{R} \qquad (1)$$

If $\vec{r}=\langle x,y,z\rangle$ and $\vec{r}_0=\langle x_0,y_0,z_0\rangle$

then,

$$\boxed{\vec{r}=\vec{r}_0+t\vec{v}}, \quad t\in\mathbb{R} \qquad (2)$$

(1) or (2) are the Line $L$ Vector equations.

Rmk: For any point $P(x,y,z)$ on $L$, there is a unique $t\in\mathbb{R}$ such that (1) is satisfied. Conversely, for any $t$ there is a unique $P(x,y,z)\in L$. 1-1.

## Parametric Form of Line L.

$\langle x,y,z\rangle = t\langle a,b,c\rangle + \langle x_0,y_0,z_0\rangle$ is equivalent to

$$\boxed{x=x_0+at, \quad y=y_0+at, \quad z=z_0+at, \quad t\in\mathbb{R}} \qquad (3)$$

## Symmetric Equations of a Line $L$ $(a,b,c \neq 0)$

From (3)

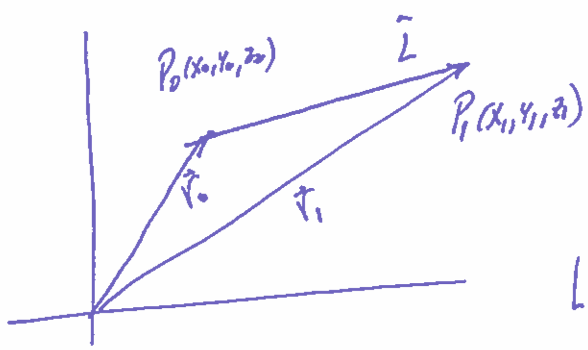
$$\boxed{t = \frac{x-x_0}{a}, \quad t = \frac{y-y_0}{b}, \quad t = \frac{z-z_0}{c} \quad \text{then} \quad \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}} \qquad (4)$$

Discuss: i) Parallel lines, ii) Perpendicular Lines

iii) Skew lines.

Show Maple worksheet on lines.

## Line segment $\hat{L}$ betw $P_0(x_0,y_0,z_0)$ and $P_1(x_1,y_1,z_1)$




$P_0 \to \vec{r}_0$

$P_1 \to \vec{r}_1$

$\overrightarrow{P_1P_0} = \vec{r}_1 - \vec{r}_0$ direction vector

Line through $P_0$ with $\overrightarrow{P_1P_0}$ as dir. vector

$$\vec{r} = \vec{r}_0 + t(\vec{r}_1 - \vec{r}_0) \Leftrightarrow \vec{r} = (1-t)\vec{r}_0 + t\vec{r}_1$$

If $t=0 \Rightarrow \vec{r} = \vec{r}_0$, if $t=1 \Rightarrow \vec{r} = \vec{r}_1$

Any point betw $P_0$ and $P_1$ can be reached for a specific $0<t<1$. Therefore,

Equ. for segment $\boxed{\vec{r} = (1-t)\vec{r}_0 + \vec{r}_1, \quad 0 \le t \le 1}$

# Planes

- Ask student for equations for planes.

Planes through $P_0$? How many?

Planes parallel to $\vec{v}$? points define a plane

Planes perpendicular to $\vec{n}$? Add one point to define a plane

Planes perpendicular to $\vec{n} = \langle a,b,c \rangle$ and passing through $P_0 \langle x_0, y_0, z_0 \rangle$?

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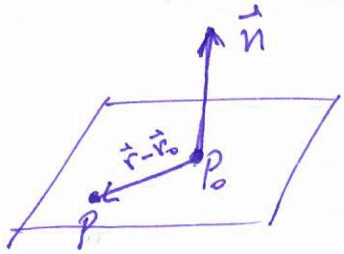
unique plane. $\vec{r} = \vec{OP}$, $\vec{r}_0 = \vec{OP_0}$




$P_0 \to \vec{r}_0$, $\vec{P_0P} = \vec{r} - \vec{r}_0$

$P \to \vec{r}$

$(\vec{r} - \vec{r}_0) \cdot \vec{n} = 0$

Therefore, $\vec{r} = \vec{OP} = \langle x, y, z \rangle$, $\vec{r}_0 = \vec{OP_0} = \langle x_0, y_0, z_0 \rangle$

$\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a, b, c \rangle = 0$

or $$\boxed{a(x-x_0) + b(y-y_0 + c(z-z_0) = 0} \quad (1)$$

Scalar equ.

or $$\boxed{ax + by + cz + d = 0} \quad (2)$$

Linear equ.

when $d = -(ax_0 + by_0 + cz_0)$

Discuss :

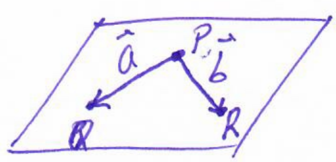
i) Parallel planes

ii) Angle between planes is the acute angle btw their normals. $\vec{n}_1, \vec{n}_2$ $\quad \cos\theta = \dfrac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|}$

iii) Perpendicular planes.

iv) Intersection of two planes. $\longrightarrow$ line $L$
or entire plane if same.

v) Equation of a plane passing through three points.



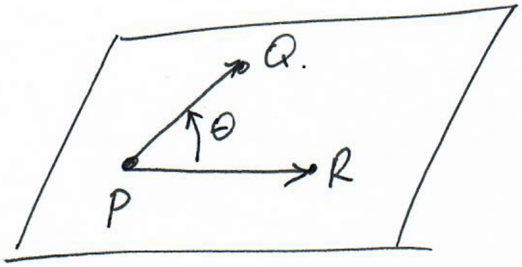

If $\vec{a} = \vec{PQ}$
$\vec{b} = \vec{PR}$

then, $\quad \vec{n} = \vec{a} \times \vec{b}$

thus, using $\vec{n}$ and $P, Q,$ or $R$ the plane eqn can be obtained.

12.5

#33) Find equ. of plane thru $P(2,1,2)$
$Q(3,-8,6)$
$R(-2,-3,1)$




$\vec{PQ} = \langle 1, -9, 4 \rangle$

$\vec{PR} = \langle -4, -4, -1 \rangle$

$$\vec{PQ} \times \vec{PR} = \vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -9 & 4 \\ -4 & -4 & -1 \end{vmatrix} = \hat{i}(25) - \hat{j}(15) + \hat{k}(-40)$$

$$= \langle 25, -15, -40 \rangle = 5\langle 5, -3, -8 \rangle$$

Using $\vec{n} = \langle 5, -3, -8 \rangle$. and $P_0(2,1,2)$.

$$\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0 \Rightarrow 5x - 3y - 8z = 10 - 3 - 16 = -9$$

$$\boxed{5x - 3y - 8z = -9}$$

12.5 #48 Line through L: $\overset{P_0}{(-3,1,0)}$ and $\overset{P_1}{(-1,5,6)}$

Find where L intersects $\pi: 2x+y-z=-2$.

Answ:

$$\vec{v} = \vec{P_0P_1} = \langle 2, 4, 6\rangle$$

$$L: \quad \langle x,y,z\rangle = t\langle 2,4,6\rangle + \langle -3,1,0\rangle.$$

$\vec{n}$ to $\pi$. $\quad \vec{n} = \langle 2,1,-1\rangle$

If $\vec{v} \perp \vec{n}$ there is not intersection.

However, $\quad \vec{v}\cdot\vec{n} = \langle 2,4,6\rangle \cdot \langle 2,1,-1\rangle = 2 \neq 0.$

So, there is an intersection.

To find it, we realize that points on L satisfy

$$x = 2t-3, \quad y = 4t+1, \quad z = 6t$$

then, for $(x,y,z)$ to be in the plane

$$2(2t-3) + 4t+1 - 6t = -2$$

$$2t - 6 + 1 = -2 \Rightarrow t = 3/2$$

and $P^*(0,7,9)$

$$\langle x^*, y^*, z^*\rangle = \frac{3}{2}\langle 2,4,6\rangle + \langle -3,1,0\rangle = \langle 3-3, 7, 9\rangle$$

$$\langle 0,7,9\rangle.$$

Problem (Similar to 12)

Find the Line of intersection of the two planes (3 ways to do it!)

$\Pi_1: x+y+z=-1, \quad \Pi_2: 2x-3y+2z=2$

Answ: For the line we need
a) a direction vector $\vec{v}=\langle a,b,c\rangle$.
b) a point on the line. $P_0(x_0,y_0,z_0)$

I a) To obtain a direction vector $\vec{v}$

$$\vec{v}=\vec{n}_1\times\vec{n}_2=\langle 1,1,1\rangle\times\langle 2,-3,2\rangle$$

or

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2 & -3 & 2 \end{vmatrix} = 5\hat{\imath}+0\hat{\jmath}-5\hat{k}=\langle 5,0,-5\rangle$$

Turn for alternative.

b) To obtain a point on L

$$x=0 \Rightarrow \begin{array}{l} (y+z=-1)(+3) \\ -3y+2z=2 \\ \hline 0\ 5z= \end{array} \Rightarrow z=-1/5 \Rightarrow y=-1-z=-4/5$$

or $\langle x_0,y_0,z_0\rangle=\langle 0,-4/5,-1/5\rangle$

Then $L: \langle x,y,z\rangle=t\langle 5,0,-5\rangle+\langle 0,-4/5,-1/5\rangle$

(II) We can also find a $2^{nd}$ point $P_2(x_2, y_2, z_2)$ at the intersect $\frac{\pi_1}{\pi_2}$ and once we have it, we can construct the line passing through

$P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$

$(0, -4/5, -1/5)$

$2^{nd}$ point: $z=0$ $(x+y=-1)(-2)$

$$2x-3y=2$$

$$-5y=4 \Rightarrow \boxed{y=-4/5} \Rightarrow \boxed{x=-1/5}$$

$$\Rightarrow P_2(x_2, y_2, z_2) = P_2(-1/5, -4/5, 0)$$

and the line at the intersection is

$$t\langle -\tfrac{1}{5}, 0, 1/5\rangle + \langle 0, -4/5, -1/5\rangle = \langle x, y, z\rangle.$$

Same as $t\langle 5, 0, -5\rangle + \langle 0, -4/5, -1/5\rangle = \langle x, y, z\rangle$

why?

$$\langle -1/5, 0, 1/5\rangle(-25) = \langle 5, 0, -5\rangle$$

either one can be used as a direction vector.

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(III) Another alternative $x=t$

$$\begin{cases} (y+z=-1-t)(-2) \\ -3y+2z = 2-2t \end{cases}$$

$$-5y=4 \Rightarrow y=-4/5$$

$$z=-1-t+4/5$$

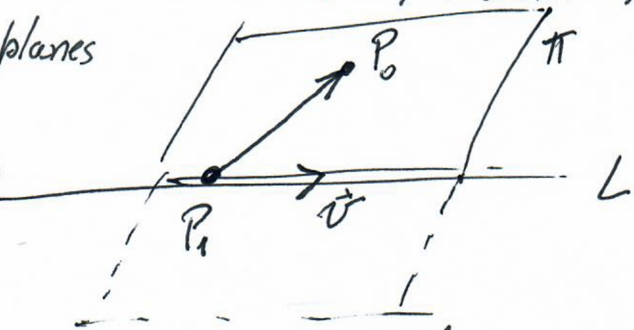
$$\langle x, y, z\rangle = \langle t, -4/5, -1-t+4/5\rangle$$

$$= t\langle 1, 0, -1\rangle + \langle 0, -4/5, -1/5\rangle.$$

12.5

**#37)** Find the plane that passes thru $P_0(3,1,4)$ and contains a line $L$, intersection of the two planes

$$\begin{cases} x+2y+3z=1 \\ 2x-y+z=-3 \end{cases}$$




$\hat{n}_{(\pi)} = \vec{v} \times \overrightarrow{P_0P_1}$

Work on this simplified problem:
Find equ. of the plane containing $L$ and $P_0$.

<u>Strategy:</u> ① obtain $\vec{v}$ using the two planes given:

$$\vec{v} = \hat{n}_1 \times \hat{n}_2$$

② Find equ. for $L$.

③ Find a point on $L$ $P_1(x_1, y_1, z_1)$ Common to both planes.

④ Form a vector $\vec{w}$ on the plane sought $\vec{w} = \overrightarrow{P_0P_1}$, $P_0(3,1,4)$.

⑤ Obtain normal $\vec{n}$ of plane sought as $\vec{n} = \vec{v} \times \overrightarrow{P_0P_1}$

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<u>answ:</u> ①
$$\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{vmatrix} = \langle 5, 5, -5 \rangle$$

② $L: \langle x, y, z \rangle = t \langle 5, 5, -5 \rangle + \underbrace{\langle 0, 2, -1 \rangle}_{= P_1}$

③ point on $L$

$x = 0 \quad 2y + 3z = 1$

$\quad 2(-y + z = -3)$

$5z = -5$

$z = -1 \Rightarrow y = -1 + 3$

$y = 2$

④ $\vec{w} = \langle 3, 1, 4 \rangle - \langle 0, 2, -1 \rangle = \langle 3, -1, 5 \rangle$

⑤
$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 5 & -5 \\ 3 & -1 & 5 \end{vmatrix} = \langle 20, -40, -20 \rangle$$
or $\langle 1, -2, -1 \rangle$

Turn

# 73) Find the distance between the parallel planes

$\pi_1 : 2x - 3y + z = 4$ and $\pi_2 : 4x - 6y + 2z = 3$

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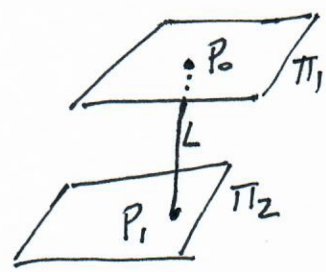
Answ: To compute the distance, we need to:

a) obtain a perpendicular line $L$ to both planes

b) Determine the two points $P_0, P_1$ in which this line $L$ intersects the two planes.




c) Compute the distance between $P_0$ and $P_1$.

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First, a point in $\pi_1$ is $\langle 0,0,4 \rangle$

Second, a Line perpendicular to $\pi_1$ and $\pi_2$ through $\langle 0,0,4 \rangle$ is (Same as constructing Line perpendicular to $\pi_2$ passing through $P_0\,(0,0,4)$)

$L: \langle x, y, z \rangle = t \langle 2,-3,1 \rangle + \langle 0,0,4 \rangle.$

Third, ~~intersection~~ with plane $\pi_2$:

Same as intersection of a given line with a plane.

$L: \; x = 2t, \quad y = -3t, \quad z = t+4,$ parametric form

Subst. into equ. for $\Pi_2$

$$4(2t) - 6(-3t) + 2(t+4) = 3$$

$$8t + 18t + 2t + 8 = 3 \Rightarrow 28t = -5 \Rightarrow \boxed{t = -5/28}$$

$$\Rightarrow \; x = \frac{-5}{14}, \quad y = \frac{15}{28}, \quad z = \frac{-5}{28} + 4 = \frac{112-5}{28} = \frac{107}{28}$$

Therefore, point of intersection of Line $L$ with plane $\Pi_2$

$$\left\langle \frac{-5}{14}, \frac{15}{28}, \frac{107}{28} \right\rangle$$

<u>Fourth</u>, distance from $\langle 0,0,4 \rangle \in \Pi_1$ to $\left\langle -\frac{5}{14}, \frac{15}{28}, \frac{107}{28} \right\rangle$ in $\Pi_2$

$$d = \sqrt{\frac{25}{(14)^2} + \frac{(15)^2}{(28)^2} + \left(\frac{107}{28} - 4\right)^2}$$

$$= \sqrt{\frac{25}{196} + \frac{225}{784} + \frac{(5)^2}{(28)^2}} = \sqrt{\frac{25}{196} + \frac{225}{784} + \frac{25}{784}}$$

$$= \sqrt{\frac{25}{196} + \frac{250}{784}} = \sqrt{\frac{25}{196} + \frac{125}{392}} = \sqrt{\frac{50+125}{392}}$$

$$= \frac{\sqrt{175}}{\sqrt{392}} = \frac{5\sqrt{7}}{2\sqrt{98}} \simeq 0.6682.$$